

# Silver block intersection graphs of Steiner 2-designs

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## Abstract

For a block design  $\mathcal{D}$ , a series of block intersection graphs  $G_i$ , or  $i$ -BIG( $\mathcal{D}$ ),  $i = 0, \dots, k$  is defined in which the vertices are the blocks of  $\mathcal{D}$ , with two vertices adjacent if and only if the corresponding blocks intersect in exactly  $i$  elements. A silver graph  $G$  is defined with respect to a maximum independent set of  $G$ , called an  $\alpha$ -set. Let  $G$  be an  $r$ -regular graph and  $c$  be a proper  $(r + 1)$ -coloring of  $G$ . A vertex  $x$  in  $G$  is said to be *rainbow* with respect to  $c$  if every color appears in the closed neighborhood  $N[x] = N(x) \cup \{x\}$ . Given an  $\alpha$ -set  $I$  of  $G$ , a coloring  $c$  is said to be *silver* with respect to  $I$  if every  $x \in I$  is rainbow with respect to  $c$ . We say  $G$  is *silver* if it admits a silver coloring with respect to some  $I$ . Finding silver graphs is of interest, for a motivation and progress in silver graphs see [7] and [15]. We investigate conditions for 0-BIG( $\mathcal{D}$ ) and 1-BIG( $\mathcal{D}$ ) of Steiner 2-designs  $\mathcal{D} = S(2, k, v)$  to be silver.

**keywords:** Silver coloring, Block intersection graph, Steiner 2-design, and Steiner triple system

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## 1 Introduction and preliminaries

We follow standard notations and concepts from design theory. For these, one may refer to, for example, [5] and [14].

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A  $2-(v, k, \lambda)$  design ( $2 < k < v$ ) is a pair  $(V, \mathcal{B})$  where  $V$  is a  $v$ -set and  $\mathcal{B}$  is a collection of  $b$   $k$ -subsets of  $V$  (blocks) such that any 2-subset of  $V$  is contained in exactly  $\lambda$  blocks. A  $2-(v, k, 1)$  design is called **Steiner 2-design** and is denoted by  $S(2, k, v)$ . An  $S(2, 3, v)$  is a **Steiner triple system** or  $\text{STS}(v)$ . A design with  $b = v$  is a **symmetric**  $(v, k, \lambda)$ -**design**. A symmetric  $S(2, k, v)$  is called a **projective plane**. If  $k$  is the size of the blocks then  $n := k - 1$  is called the order of the plane. This design is usually denoted by  $\text{PG}(2, n)$ . A  $2-(n^2, n, 1)$  design is called an **affine plane**. For such design we use the notation  $\text{AG}(2, n)$ .

A **partial parallel class** is a set of blocks that contains no element of the design more than once. A **parallel class** (PC) or a **resolution class** in a design is a set of blocks that partition the set of elements  $V$ . A **near parallel class** is a partial parallel class missing a single element. A **resolvable balanced incomplete block design** is a  $2-(v, k, \lambda)$  design whose blocks can be partitioned into parallel classes. The notation  $\text{RBIBD}(v, k, \lambda)$  is commonly used. An affine plane of order  $n$  is an  $\text{RBIBD}(n^2, n, 1)$ . A resolvable  $\text{STS}(v)$  together with a resolution of its blocks is called a **Kirkman triple system**,  $\text{KTS}(v)$ .

Given a design  $\mathcal{D}$ , a series of **block intersection graphs**  $G_i$ , or  $i$ -BIG,  $i = 0, \dots, k$  can be defined in which the vertices are the blocks of  $\mathcal{D}$ , with two vertices are adjacent if and only if the corresponding blocks intersect in exactly  $i$  elements.

**Example 1** For  $\text{STS}(7)$ , 0-BIG is empty graph and 1-BIG is  $K_7$ . For  $\text{STS}(9)$ , 0-BIG is disconnected and consists of four disjoint  $K_3$ 's and 1-BIG is  $K_{3,3,3,3}$ .

The study of  $i$ -BIG( $\mathcal{D}$ ) is useful in characterizing block designs. Some researchers have studied properties of various kinds of block intersection graphs, see for example [1], [2], [4], [8], [9], [10], [16], and [17].

A graph of order  $v$  is **strongly regular**, denoted by  $\text{SRG}(v, k, \lambda, \mu)$ , whenever it is not complete or edgeless and, (i) each vertex is adjacent to  $k$  vertices, (ii) for each pair of adjacent vertices there are  $\lambda$  vertices adjacent to both, (iii) for each pair of non-adjacent vertices there are  $\mu$  vertices adjacent to both.

**Remark 1** Let  $G_i$  be the  $i$ -block intersection graph of an  $S(2, k, v)$ . Then for each  $i = 2, 3, \dots, k$ , the graph  $G_i$  is empty. So we consider only  $G_0$  and  $G_1$ . Graphs  $G_0$  and  $G_1$  are complements of each other.  $G_1$  is an  $\text{SRG}(b, k(r - 1), r - 2 + (k - 1)^2, k^2)$  and  $G_0$  is an  $\text{SRG}(b, b - k(r - 1) - 1, b - 2k(r - 1) + k^2 - 2, b - 2kr + k^2 + r - 1)$  (see Chapter 21 of [14]).

In a graph  $G = (V, E)$  an **independent set** is a subset of vertices no two of which are adjacent. The **independence number**  $\alpha(G)$  is the cardinality of a largest set of independent vertices. We refer to any maximum independent set of a graph as an  $\alpha$ -set. Let  $c$  be a proper  $(r + 1)$ -coloring of an  $r$ -regular graph  $G$ . A vertex  $x$  in  $G$  is said to be **rainbow** with respect to  $c$  if

every color appears in the closed neighborhood  $N[x] = N(x) \cup \{x\}$ . Given an  $\alpha$ -set  $I$  of  $G$  the coloring  $c$  is said to be **silver** with respect to  $I$  if every  $x \in I$  is rainbow with respect to  $c$ . We say  $G$  is silver if it admits a silver coloring with respect to some  $\alpha$ -set. If all vertices of  $G$  are rainbow, then  $c$  is called a **totally silver** coloring of  $G$  and  $G$  is said to be totally silver. Note that the definition of silver coloring depends on the chosen  $\alpha$ -set. For example in Figure 1, a graph  $G$  is shown which is silver when the  $\alpha$ -set (the bold vertices) is taken as in the left, but it does not have any silver coloring with the  $\alpha$ -set taken as on the right hand side.



Figure 1: A silver coloring of a graph

There are many different version of rainbow colorings in the literature, for example see [3], [11], [12], and [13]. For a motivation and progress in silver graphs see [7] and [15]. In fact silver graphs are closely related to a concept in graph coloring, called defining set. Let  $c$  be a proper  $k$ -coloring of a graph  $G$  and let  $S \subseteq V(G)$ . If  $c$  is the only extension of  $c|_S$  to a proper  $k$ -coloring of  $G$ , then  $S$  is called a **defining set** of  $c$ . The minimum size of a defining set among all  $k$ -colorings of  $G$  is called a **defining number** and denoted by  $\text{def}(G, k)$ . A more general survey of defining sets in combinatorics appears in [6]. Let  $G$  be an  $r$ -regular graph, then  $G$  is silver if and only if  $\text{def}(G, r + 1) = |V(G)| - \alpha(G)$ . In [15] an open problem is raised:

**Question 1** *Find classes of  $r$ -regular graphs  $G$ , for which  $\text{def}(G, r + 1) = |V(G)| - \alpha(G)$ , i.e. determine classes of all silver graphs.*

A silver cube is a silver graph  $G = K_n^d$ , the Cartesian power of the complete graph  $K_n$ . Silver cubes are generalizations of silver matrices, which are  $n \times n$  matrices where each symbol in  $\{1, 2, \dots, 2n - 1\}$  appears in either the  $i$ -th row or the  $i$ -th column of the matrix. In [7] some algebraic constructions and a product construction of silver cubes are given. They show the relation of these cubes to codes over finite fields, dominating sets of a graph, Latin squares, and finite geometry. In particular the Hamming codes are used to produce a totally silver cube and the bound for the best binary codes is used to prove the non-existence of silver cubes for a large class of parameters with  $n = 2$ .

To study Question 1, here we consider  $i$ -BIGs of designs. First we give some examples of designs with silver  $i$ -BIGs.

**Example 2** In any symmetric  $(v, k, \lambda)$ -design  $\mathcal{D}$ , every two distinct blocks have exactly  $\lambda$  elements in common, so for  $0 \leq i \leq k$ ,  $i \neq \lambda$ ,  $i$ -BIG( $\mathcal{D}$ ) is empty graph, and  $\lambda$ -BIG( $\mathcal{D}$ ) is complete graph. Hence all of these graphs are totally silver. Specifically for each  $k$  and  $0 \leq i \leq k+1$ ,  $i$ -BIG( $S(2, k+1, k^2+k+1)$ ) is totally silver.

If  $\mathcal{D}$  is an AG(2,  $n$ ), then  $G_0 = 0$ -BIG( $\mathcal{D}$ ) consists of  $(n+1)$  disjoint  $K_n$ 's, so it is totally silver, and  $G_1 = 1$ -BIG( $\mathcal{D}$ ) =  $\underbrace{K_{n, n, \dots, n}}_{n+1}$  is silver.

In this paper we prove the following results: If an  $S(2, k, v)$  contains a parallel class, then a necessary condition for 1-BIG( $S(2, k, v)$ ) to be silver is  $k^2 \mid v$ . For each admissible  $v = 9m$  we construct a  $\mathcal{D}_1 = \text{KTS}(v)$ , such that 1-BIG( $\mathcal{D}_1$ ) is silver. And in general for each  $k$  and  $v$  where an AG(2,  $k$ ) and an RBIBD( $v, k, 1$ ) exist we construct a  $\mathcal{D}^* = \text{RBIBD}(kv, k, 1)$  such that 1-BIG( $\mathcal{D}^*$ ) is silver. Also a lower bound for  $\alpha(G_1)$  is given in order for a 1-BIG( $S(2, k, v)$ ) to be silver. For any admissible  $v$ , the existence of a silver 1-BIG( $S(2, k, v)$ ) which possesses a maximum possible independent set, i.e. of size  $\frac{v}{k}$  or  $\frac{v-1}{k}$ , is settled. We prove that for  $v > k^3 - 2k^2 + 2k$  there is no silver 0-BIG( $S(2, k, v)$ ). Also we settle the question of existence of silver 0-BIG(STS( $v$ )) for all admissible  $v$ .

Since every vertex of  $i$ -BIG( $\mathcal{D}$ ) corresponds to a block of  $\mathcal{D}$ , we will mostly refer to them as “blocks” rather than vertices. The following notation will be used in our discussion. Let  $G$  be a graph and  $I$  be an  $\alpha$ -set of  $G$ . For each  $i = 1, \dots, |I|$ , we let

$$X_i := \{u \mid u \in V(G) \setminus I, u \text{ is adjacent to exactly } i \text{ vertices of } I\}.$$

## 2 One block intersection graphs

The following is a necessary condition for 1-BIG( $\mathcal{D}$ ) of a Steiner system  $\mathcal{D} = S(2, k, v)$  with  $\alpha(G_1) = \frac{v}{k}$ , to be silver.

**Theorem 1** Let  $\mathcal{D}$  be an  $S(2, k, v)$ , which has a parallel class, and let  $G_1$  be 1-BIG( $\mathcal{D}$ ). A necessary condition for  $G_1$  to be silver is  $k^2 \mid v$ .

**Proof.**  $G_1$  is a  $\frac{k(v-k)}{(k-1)}$ -regular graph. Let  $I$  be an  $\alpha$ -set, and assume that  $G_1$  has a silver coloring with respect to  $I$  with  $C$  as the set of colors. We have  $|I| = \frac{v}{k}$ , and  $|C| = \frac{k(v-k)}{k-1} + 1$ . Since  $|C| > |I|$ , a color like  $\iota$  exists that is not used in  $I$ . The vertices of  $I$  are rainbow, and each vertex with color  $\iota$  from  $V(G_1) \setminus I$ , must be adjacent to  $k$  distinct vertices of  $I$ . Therefore  $|I|$  must be a multiple of  $k$ , which implies  $k^2 \mid v$ . ■

**Example 3** *There are 80 nonisomorphic STS(15)s, where 70 of them have parallel class (see [5], page 32). So by Theorem 1, none of those 70 has silver  $G_1$ .*

By Theorem 1, if  $v$  is not a multiple of 9, then no silver 1-BIG(KTS( $v$ )) exists. In the next lemma we show that for the case  $9 \mid v$ , when a KTS( $v$ ) exists, i.e.  $v = 18q + 9$ , there exists a silver 1-BIG(KTS( $v$ )). This lemma is an illustration of a general structure which will be discussed in Theorem 2.

**Lemma 1** *If  $v \equiv 3 \pmod{6}$ , then a  $\mathcal{K} = \text{KTS}(3v)$  exists such that 1-BIG( $\mathcal{K}$ ) is silver.*

**Proof.** Let  $\mathcal{A} = \text{AG}(2, 3) = \text{STS}(9)$  with  $V(\mathcal{A}) = \{(i, j) \mid 1 \leq i, j \leq 3\}$ , and denote its parallel classes by:

$$\begin{array}{ccc}
 & \Theta_0 & \\
 & \{(1, 1), (2, 1), (3, 1)\} \\
 & \{(1, 2), (2, 2), (3, 2)\} \\
 & \{(1, 3), (2, 3), (3, 3)\} \\
 \\
 \Theta_1 & \Theta_2 & \Theta_3 \\
 a_1 = \{(1, 1), (1, 2), (1, 3)\} & a_4 = \{(1, 1), (2, 2), (3, 3)\} & a_7 = \{(1, 1), (2, 3), (3, 2)\} \\
 a_2 = \{(2, 1), (2, 2), (2, 3)\} & a_5 = \{(1, 3), (2, 1), (3, 2)\} & a_8 = \{(1, 2), (2, 1), (3, 3)\} \\
 a_3 = \{(3, 1), (3, 2), (3, 3)\} & a_6 = \{(1, 2), (2, 3), (3, 1)\} & a_9 = \{(1, 3), (2, 2), (3, 1)\}
 \end{array}$$

Consider a KTS( $v$ )  $\mathcal{D} = (V, \mathcal{B})$ ,  $V = \{x_1, x_2, \dots, x_v\}$  with parallel classes  $\pi_1, \pi_2, \dots, \pi_{\frac{v-1}{2}}$ . Using its blocks we construct  $\mathcal{K} = (V^*, \mathcal{B}^*)$ , a KTS( $3v$ ) in the following manner.

The set of elements of  $\mathcal{K}$  is  $V^* = \{1, 2, 3\} \times V$ , and the blocks are introduced in the following 4 types of parallel classes,  $\Omega_{0,\beta}$ ,  $\Omega_{1,\beta}$ ,  $\Omega_{2,\beta}$  and  $\Omega_{3,\beta}$ .

- $\Omega_{0,\beta} : \left\{ \{(1, x_i), (2, x_i), (3, x_i)\} \mid 1 \leq i \leq v \right\}$ .

We denote every block of  $\mathcal{D}$  by  $\{x_i, x_j, x_k\}$ , where  $i < j < k$ . In the following a label  $(m, \beta)$  for each block is its color, the block with label  $(m, \beta)$  is obtained by using the block  $a_m$  of  $\mathcal{A}$ .

- $\Omega_{1,\beta} : \left\{ \{(1, x_i), (1, x_j), (1, x_k)\}_{(1,\beta)}, \{(2, x_i), (2, x_j), (2, x_k)\}_{(2,\beta)}, \{(3, x_i), (3, x_j), (3, x_k)\}_{(3,\beta)} \mid \{x_i, x_j, x_k\} \in \pi_\beta \right\}, \text{ for } 1 \leq \beta \leq \frac{v-1}{2},$
- $\Omega_{2,\beta} : \left\{ \{(1, x_i), (2, x_j), (3, x_k)\}_{(4,\beta)}, \{(1, x_k), (2, x_i), (3, x_j)\}_{(5,\beta)}, \{(1, x_j), (2, x_k), (3, x_i)\}_{(6,\beta)} \mid \{x_i, x_j, x_k\} \in \pi_\beta \right\}, \text{ for } 1 \leq \beta \leq \frac{v-1}{2},$

- $\Omega_{3,\beta}$ :  $\left\{ \{(1, x_i), (2, x_k), (3, x_j)\}_{(7,\beta)}, \{(1, x_j), (2, x_i), (3, x_k)\}_{(8,\beta)}, \{(1, x_k), (2, x_j), (3, x_i)\}_{(9,\beta)} \mid \{x_i, x_j, x_k\} \in \pi_\beta \right\}$ , for  $1 \leq \beta \leq \frac{v-1}{2}$ .

Figures 2 and 3 demonstrate the 4 types of blocks.

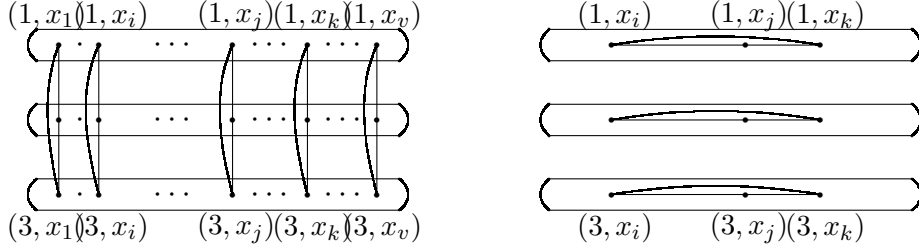


Figure 2: Blocks of  $\Omega_{0,\beta}$  and  $\Omega_{1,\beta}$

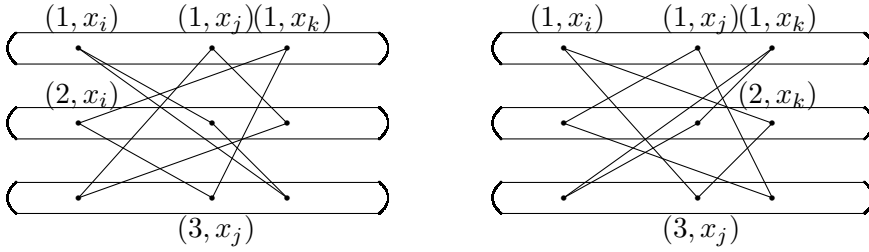


Figure 3: Blocks of  $\Omega_{2,\beta}$  and  $\Omega_{3,\beta}$

We note that there is only one parallel class in  $\Omega_{0,\beta}$ , but there are  $\frac{v-1}{2}$  parallel classes in each of other types, so we have  $\frac{3v-1}{2}$  parallel classes and each class has  $v$  blocks.

Clearly,  $\mathcal{K}$  is a  $\text{KTS}(3v)$ . The number of colors needed in a silver coloring of  $1\text{-BIG}(\mathcal{K})$  is equal to  $\frac{9v-7}{2}$ . We color 0 the vertices corresponding to the blocks in  $\Omega_{0,\beta}$  class. The label of each block in other classes, which is shown as its index, is the color of its corresponding vertex in  $1\text{-BIG}(\mathcal{K})$ :  $(m, \beta)$ ,  $1 \leq m \leq 9$ ,  $1 \leq \beta \leq \frac{v-1}{2}$ . It is easy to check that this is a proper coloring and all vertices in  $\Omega_{0,\beta}$  class, i.e. the  $\alpha$ -set, are rainbow. ■

Next theorem is a generalization of the construction introduced in Lemma 1.

**Theorem 2** *Assume there exist an affine plane  $\mathcal{A} = \text{AG}(2, k)$ , and a resolvable balanced incomplete block design  $\mathcal{D} = \text{RBIBD}(v, k, 1)$ . Then there exists a  $\mathcal{D}^* = \text{RBIBD}(kv, k, 1)$  where  $1\text{-BIG}(\mathcal{D}^*)$  is silver.*

**Proof.** Let  $V(\mathcal{A}) = \{(i, j) \mid 1 \leq i, j \leq k\}$  and denote its parallel classes by  $\Theta_0, \Theta_1, \dots, \Theta_k$ . Specifically we let

$$\Theta_0 = \left\{ \{(1, j), (2, j), \dots, (k, j)\} \mid j = 1, 2, \dots, k \right\}.$$

Also we let  $V(\mathcal{D}) = \{x_1, x_2, \dots, x_v\}$  with parallel classes  $\pi_1, \pi_2, \dots, \pi_{\frac{v-1}{k-1}}$ .

For each block  $b = \{x_{s_1}, x_{s_2}, \dots, x_{s_k}\}$  of  $\mathcal{D}$  we consider an ordering on  $b$  such that

$$x_{s_i} \prec x_{s_j} \iff s_i < s_j,$$

and define a function:

$$\Psi_b : V(\mathcal{A}) \rightarrow \{1, 2, \dots, k\} \times \{x_{s_1}, x_{s_2}, \dots, x_{s_k}\}$$

$$\Psi_b(i, j) = (i, x_{s_j}).$$

We extend  $\Psi_b$  for each block  $a$  of  $\mathcal{A}$  as  $\Psi_b(a) = \{\Psi_b(i, j) \mid (i, j) \in a\}$ .

Now we construct a design  $\mathcal{D}^* = (V^*, \mathcal{B}^*)$ , as in the following:

$$V^* = \{1, 2, \dots, k\} \times V(\mathcal{D}).$$

$$\mathcal{B}^* = \{\Psi_b(a) \mid b \text{ and } a \text{ are blocks of } \mathcal{D} \text{ and } \mathcal{A}, \text{ respectively}\}.$$

See Figure 4.

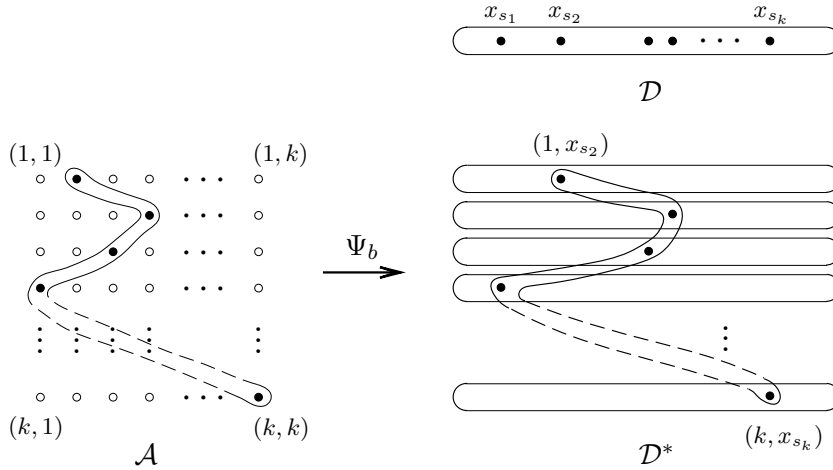


Figure 4: Blocks of  $\mathcal{D}^*$  are constructed by using blocks of  $\mathcal{A}$

$\mathcal{D}^*$  is an RBIBD with the following parallel classes:

$$\Omega_{\alpha, \beta} = \{\Psi_b(a) \mid a \in \Theta_\alpha, b \in \pi_\beta\}, \text{ for each } 0 \leq \alpha \leq k \text{ and } 1 \leq \beta \leq \frac{v-1}{k-1}.$$

Note that:

$$\Omega_{0,1} = \Omega_{0,2} = \dots = \Omega_{0, \frac{v-1}{k-1}} = \left\{ \{(1, x_s), (2, x_s), \dots, (k, x_s)\} \mid s = 1, 2, \dots, v \right\}.$$

We show that  $1\text{-BIG}(\mathcal{D}^*)$  is silver with respect to the  $\alpha$ -set

$$\begin{aligned} I^* &= \{\Psi_b(a) \mid a \in \Theta_0 \text{ and } b \text{ is a block of } \mathcal{D}\} \\ &= \left\{ \{(1, x_s), (2, x_s), \dots, (k, x_s)\} \mid s = 1, 2, \dots, v \right\}, \end{aligned}$$

by the following coloring:

$$c : \mathcal{B}^* \longrightarrow \{0\} \cup \{(a, \beta) \mid a \text{ is a block of } \mathcal{A} \setminus \Theta_0 \text{ and } 1 \leq \beta \leq \frac{v-1}{k-1}\}$$

$$\Psi_b(a) \longmapsto \begin{cases} 0 & \text{if } a \in \Theta_0, \\ (a, \beta) & \text{if } a \notin \Theta_0, \text{ and } b \in \pi_\beta. \end{cases}$$

We show that  $c$  is a proper coloring and any vertex  $b^* \in I^*$  is rainbow. Note that all the vertices of  $I^*$  have color 0. Let  $\Psi_{b_1}(a_1)$  and  $\Psi_{b_2}(a_2)$  be two blocks of  $\mathcal{D}^*$  with the same color  $(a, \beta)$ . Then we have  $b_1, b_2 \in \pi_\beta$ . Therefore  $b_1$  and  $b_2$  are disjoint blocks of  $\mathcal{D}$ , so  $\Psi_{b_1}(a_1)$  and  $\Psi_{b_2}(a_2)$  are disjoint. Thus  $c$  is proper.

To show silverness, for a fixed  $s$  let  $b_s^* = \{(1, x_s), (2, x_s), \dots, (k, x_s)\}$  be a block of  $I^*$ . By definition, for any given nonzero color like  $(a, \beta)$  we have  $a \notin \Theta_0$ , and there exists a unique block  $b$  of  $\pi_\beta$  which contains  $x_s$  and the color of  $\Psi_b(a)$  is  $(a, \beta)$ . Since in  $\mathcal{A}$ , the block  $a$  intersects each block of  $\Theta_0$ , thus by definition of  $\mathcal{B}^*$ ,  $\Psi_b(a)$  intersects  $b_s^*$  in  $\mathcal{D}^*$ , so the color  $(a, \beta)$  appears in the neighborhood of  $b_s^*$ . ■

In the next theorem for any  $\mathcal{D} = S(2, k, v)$ , we show a lower bound for  $\alpha(G_1)$ , in order  $G_1 = 1\text{-BIG}(\mathcal{D})$  to be silver.

**Theorem 3** *Let  $\mathcal{D}$  be an  $S(2, k, v)$ , and  $G_1 = 1\text{-BIG}(\mathcal{D})$ . If  $\alpha(G_1) > k \lfloor \frac{v(v-1)}{k^2v-k^3+k^2-k} \rfloor$ , then  $G_1$  is not silver.*

**Proof.**  $G_1$  is a  $\frac{k(v-k)}{(k-1)}$ -regular graph with  $\frac{v(v-1)}{k(k-1)}$  vertices. Let  $I$  be an  $\alpha$ -set, and assume that  $G_1$  has a silver coloring with respect to  $I$  with  $C$  as the set of colors,  $|C| = \frac{k(v-k)}{k-1} + 1$ . A color like  $\iota$  exists that is used in the coloring of at most  $\lfloor \frac{|V(G_1)|}{|C|} \rfloor = \lfloor \frac{v(v-1)}{k^2v-k^3+k^2-k} \rfloor$  vertices of  $G_1$ . For a set  $X \subseteq V(G_1)$  we denote the set of vertices with color  $\iota$  in  $X$  by  $X(\iota)$ . By counting the number of appearances of color  $\iota$  in  $I$  and in the neighborhood of  $I$  we obtain,

$$\begin{aligned} \alpha(G_1) &= |I(\iota)| + |X_1(\iota)| + 2|X_2(\iota)| + \dots + k|X_k(\iota)| \\ &\leq k(|I(\iota)| + |X_1(\iota)| + |X_2(\iota)| + \dots + |X_k(\iota)|) \\ &\leq k \lfloor \frac{v(v-1)}{k^2v-k^3+k^2-k} \rfloor \\ &< \alpha(G_1). \end{aligned}$$

A contradiction. ■

**Example 4** *It is easy to check that for any of two STS(13)s,  $\alpha(G_1) = 4$ . For 80 nonisomorphic STS(15)s, we have  $\alpha(G_1) = 4$  or 5 (see [5], page 32). Also there are 18 nonisomorphic  $S(2, 4, 25)$  (see [5], page 34), by a computer search they have  $\alpha(G_1) = 5$  or 6. So by Theorem 3 none of them has a silver  $G_1$ .*



**Remark 2** Let  $G_1$  be the 1-block intersection graph of an  $S(2, k, v)$  with a parallel class. Then  $\alpha(G_1) = \frac{v}{k}$ , and all the elements of  $V$  appear in the blocks corresponding to each  $\alpha$ -set. Let  $I$  be an  $\alpha$ -set for  $G_1$ , therefore any vertex of  $V(G_1) \setminus I$  is adjacent to  $k$  vertices of  $I$ . Thus  $|X_1| = |X_2| = \dots = |X_{k-1}| = 0$ ,  $|X_k| = \frac{v(v-k)}{k(k-1)}$ .

If an  $S(2, k, v)$  has a near parallel class, then  $\alpha(G_1) = \frac{v-1}{k}$ , and each  $\alpha$ -set contains all the elements of  $V$  except one. Hence in this case any vertex of  $V(G_1) \setminus I$  is adjacent to either  $(k-1)$  or  $k$  vertices of  $I$ , and  $|X_1| = |X_2| = \dots = |X_{k-2}| = 0$ ,  $|X_{k-1}| = \frac{v-1}{k-1}$ ,  $|X_k| = \frac{(v-1)(v-2k+1)}{k(k-1)}$ .

**Theorem 4** Let  $\mathcal{D}$  be an  $S(2, k, v)$ , with a near parallel class. Then  $G_1 = 1\text{-BIG}(\mathcal{D})$  is not silver.

**Proof.** Let  $I$  be an  $\alpha$ -set for  $G_1$ . Assume that  $G_1$  has a silver coloring with respect to  $I$  and  $C$  is the set of colors.  $G_1$  is  $\frac{k(v-k)}{k-1}$ -regular,  $|C| = \frac{k(v-k)}{k-1} + 1$  and  $|I| = \frac{v-1}{k}$ . By Remark 2,  $|X_{k-1}| = \frac{v-1}{k-1}$  and  $|X_k| = \frac{(v-1)(v-2k+1)}{k(k-1)}$ . Since  $|C| > |I \cup X_{k-1}|$ , a color like  $\iota$  exists that is used only in the coloring of vertices of  $X_k$ . The vertices of  $I$  are rainbow, so each of the vertices of  $X_k$  that have color  $\iota$ , must be adjacent to  $k$  different vertices of  $I$ . Thus  $|I|$  is a multiple of  $k$ , say  $|I| = mk$ .

Since  $|X_{k-1}| = \frac{v-1}{k-1} > |I|$ , a color like  $\iota'$  exists that is used in the coloring of vertices of  $X_{k-1}$  but is not used in  $I$ . The induced subgraph on  $X_{k-1}$  is a clique, so  $\iota'$  appears only in one vertex of  $X_{k-1}$  and it has  $(k-1)$  neighbors in  $I$ . Thus  $|I| - k + 1$  vertices of  $I$ , each must have a neighbor in  $X_k$  with color  $\iota'$ . Again vertices from  $X_k$  that have color  $\iota'$ , each must be adjacent to  $k$  different vertices of  $I$ . Therefore  $|I| - k + 1 = (m-1)k + 1$  is also a multiple of  $k$ . This is impossible. ■

**Example 5** The 1-block intersection graph of any Hanani triple system (see [5], page 67 for the definition) is not silver.

Note that by Theorems 1, 2, 3, and 4, for any admissible  $v$  the problem of existence of a silver 1-BIG( $S(2, k, v)$ ) which possesses maximum possible independent set is settled.

### 3 Zero block intersection graphs

In this section we discuss 0-block intersection graphs of  $S(2, k, v)$ .

**Notation 1** Let  $x$  be a given element of  $S(2, k, v)$ , and denote by  $T(x)$  the set of  $\frac{v-1}{k-1}$  blocks containing  $x$ .

It is trivial that  $T(x)$  is an independent set for  $G_0$ , thus  $\alpha(G_0) \geq \frac{v-1}{k-1}$ .

**Lemma 2** Let  $\mathcal{D}$  be an  $S(2, k, v)$ , and  $G_0 = 0\text{-BIG}(\mathcal{D})$ . If  $v > k^3 - 2k^2 + 2k$  then any maximum independent set of  $G_0$  is of the form  $T(x)$ , therefore  $\alpha(G_0) = \frac{v-1}{k-1}$ .

**Proof.** Let  $I$  be an  $\alpha$ -set of  $G_0$ . Suppose  $I$  is not of the form  $T(x)$ . There exists an element  $x_0$  of  $\mathcal{D}$  which appears in at least two blocks of  $I$ . Let  $I_1 = \{B_1, B_2, \dots, B_p\} = \{B \mid B \in I \cap T(x_0)\}$ , and  $I \setminus I_1 = \{B_{p+1}, B_{p+2}, \dots, B_{p+q}\}$ . Since  $\lambda = 1$ , for  $1 \leq i < j \leq p$ ,  $(B_i \setminus \{x_0\}) \cap (B_j \setminus \{x_0\}) = \emptyset$ . Every two blocks in  $I$  have one intersection. So, for each block  $B \in I \setminus I_1$  we have  $B \cap B_i = \{a_i\}$ ,  $i = 1, 2, \dots, p$ . So  $p \leq |B| = k$ .

Now suppose  $B_1, B_2 \in I_1$ . There exist exactly  $(k-1)^2$  pairs  $\{x, y\}$  where  $x \in B_1 \setminus \{x_0\}$  and  $y \in B_2 \setminus \{x_0\}$ , and each of these pairs appears at most in one of the blocks of  $I \setminus I_1$ . Thus  $q \leq (k-1)^2$ .

So  $|I| = p + q \leq k + (k-1)^2$ . But since  $v > k^3 - 2k^2 + 2k$ , for each  $x$  we have  $|T(x)| = \frac{v-1}{k-1} > k + (k-1)^2 \geq |I|$ . Hence the statement follows. ■

**Theorem 5** Let  $\mathcal{D}$  be an  $S(2, k, v)$ . For  $v > k^3 - 2k^2 + 2k$ ,  $G_0 = 0\text{-BIG}(\mathcal{D})$  is not silver.

**Proof.**  $G_0$  is a  $\frac{v^2+k^3-v(k^2+1)-k^2+k}{k(k-1)}$ -regular graph (Remark 1). Let  $I$  be any  $\alpha$ -set for  $G_0$ . By Lemma 2,  $I = T(x)$  and  $|I| = \alpha(G_0) = \frac{v-1}{k-1}$ . Since each block out of  $I$  intersects exactly  $k$  blocks of  $I$ , each vertex of  $V(G_0) \setminus I$  is adjacent to  $\frac{v-1}{k-1} - k = \frac{v-1-k^2+k}{k-1}$  vertices of  $I$ . Then  $V(G_0) = I \cup X_{\frac{v-1-k^2+k}{k-1}}$  and  $|X_{\frac{v-1-k^2+k}{k-1}}| = \frac{(v-1)(v-k)}{k(k-1)}$ .

To the contrary,  $G_0$  has a silver coloring with respect to  $I$ . Let  $C$  be the set of colors,  $|C| = \frac{v^2-v-k^2v+k^3}{k(k-1)}$ . Since  $|C| > \frac{v-1}{k-1}$ , a color like  $\iota$  exists that is not used in the coloring of  $I$ . The vertices of  $I$  are rainbow, and the vertices from  $X_{\frac{v-1-k^2+k}{k-1}}$  that have color  $\iota$ , each must be adjacent to  $\frac{v-1-k^2+k}{k-1}$  different vertices of  $I$ . Therefore  $|I|$  must be divisible by  $\frac{v-1-k^2+k}{k-1}$ , then  $(v - k^2 + k - 1) \mid (v - 1)$  which is impossible, since  $v > k^3 - 2k^2 + 2k$ . Therefore graph  $G_0$  is not silver with respect to any  $\alpha$ -set. ■

### 3.1 0-BIG for Steiner triple systems

Both  $0\text{-BIG}(\text{STS}(v))$  for  $v = 7$  and  $v = 9$ , by Example 2, are totally silver.

**Theorem 6** For any admissible  $v > 9$ ,  $G_0 = 0\text{-BIG}(\text{STS}(v))$  is not silver.

**Proof.** For  $v > 15$ , it follows by Theorem 5.

If  $v \leq 15$ , then suppose  $I$  is an  $\alpha$ -set of  $G_0$ , and  $I$  is not of the form  $T(x)$ . Then it is easy to check that, each element of  $\text{STS}(v)$  appears at most in 3 blocks of  $I$ . If it has 3 blocks containing an element  $x$ , then such a set has at most 7 blocks, and they are contained in  $I_1$ , where:

$$I_1 = \{\{x, a, b\}, \{x, c, d\}, \{x, e, f\}, \{a, c, f\}, \{a, d, e\}, \{b, c, e\}, \{b, d, f\}\} \approx \text{STS}(7).$$

Now we discuss possible cases.

$v = 15$ :

For  $v = 15$  an  $\alpha$ -set,  $I$ , may be of the form  $T(x)$  or it may come from a subsystem  $\text{STS}(7)$ , in either case  $\alpha(G_0) = 7$ . From 80 non-isomorphic  $\text{STS}(15)$ s, 23 of them have a subsystem  $\text{STS}(7)$  ([5], page 32). It is straightforward to check that in all of  $\text{STS}(15)$ s for any  $\alpha$ -set  $I$ , each block out of  $I$  has intersection with exactly three blocks of  $I$ . So each vertex in  $V(G_0) \setminus I$  is adjacent to exactly four vertices of  $I$ . In any silver coloring with  $C$  as the set of colors of  $G_0$ , we have  $|C| = 17 > 7 = |I|$ . So there exists a color  $\iota$  which is not used in  $I$ . Every vertex with the color  $\iota$  has exactly 4 neighbors in  $I$ , therefore 7 must be a multiple of 4. So  $G_0$  does not have a silver coloring.

$v = 13$ :

For  $v = 13$  there are two non-isomorphic  $\text{STS}(13)$ s. No  $\text{STS}(13)$  has a subsystem of  $\text{STS}(7)$ , even no  $\text{STS}(13)$  has 6 blocks of an  $\text{STS}(7)$ . So, in  $G_0$  for both of them, the sets of the form  $T(x)$ , are the only  $\alpha$ -sets and  $\alpha(G_0) = 6$ . Suppose  $I$  is any  $\alpha$ -set.

First, we show that it is always possible to find three vertices in  $I$  with no common neighbor:

- One of two  $\text{STS}(13)$ s, Type 1, has a cyclic automorphism, and we can construct its blocks on  $\{1, 2, \dots, 13\}$  by the following base blocks:

$$\{1, 2, 5\}, \quad \{1, 3, 8\} \quad \text{mod } 13.$$

If  $I = T(1)$ , then  $B_1 = \{1, 2, 5\}$ ,  $B_2 = \{1, 3, 8\}$ , and  $B_3 = \{1, 10, 11\}$  do not have common neighbor. Let  $x \neq 1$  be a given element of  $\text{STS}(v)$ , and  $I = T(x)$ . Three vertices of  $I$ ,  $B'_1, B'_2, B'_3$  are obtained by adding  $(x - 1)$  to all members of blocks  $B_1, B_2, B_3$ , do not have common neighbor.

- The other  $\text{STS}(13)$  is non-cyclic and we can construct its blocks from Type 1 by

replacing four blocks of trade  $T_1$  with four blocks of trade  $T_2$  as follows:

$$\begin{array}{rcc}
 & 1 & 2 & 5 \\
 T_1 : & 1 & 3 & 8 \\
 & 10 & 2 & 8 \\
 & 10 & 3 & 5
 \end{array}
 \qquad
 \begin{array}{rcc}
 & 1 & 2 & 8 \\
 T_2 : & 1 & 3 & 5 \\
 & 10 & 2 & 5 \\
 & 10 & 3 & 8
 \end{array}$$

Let  $I = T(x)$  for some  $x$ . If  $x$  is an element of  $T_2$ , i.e.  $x \in \{1, 2, 3, 5, 8, 10\}$ , then there are two blocks say  $B_1$  and  $B_2$  of  $T_2$  which contain  $x$ . There exists one element  $y$ , such that  $y \in T_2$  but  $y \notin B_1 \cup B_2$ . We consider  $B_3$ , the block containing  $x$  and  $y$ . Then these three blocks do not have common neighbor. If  $x$  is not in  $T_2$ , then we consider several cases for  $I = T(x)$ , and show that there exist three vertices of  $I$ , which do not have common neighbor.

Now, assume for some STS(13),  $G_0 = 0\text{-BIG}(\text{STS}(13))$  is silver with respect to some  $\alpha$ -set  $I = T(x) = \{B_1, B_2, B_3, B_4, B_5, B_6\}$ . The color of all neighbors of  $B_i$ ,  $i = 1, \dots, 6$ , must be distinct. Assume  $\{B_1, B_2, B_3\} \subset I$  do not have common neighbor. Let  $N(B_i)$  be the set of neighbors of  $B_i$ .  $G_0 = \text{SRG}(26, 10, 3, 4)$ , so  $|N(B_1) \cap N(B_2)| + |N(B_2) \cap N(B_3)| + |N(B_1) \cap N(B_3)| = 12$ . Thus the color of these vertices must be distinct, while we have only 11 colors. Therefore  $G_0$  does not have a silver coloring. ■

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